MATH 183 - Final Exam
December 08, 2014

Name:____

1. Let $f(x, y) = y^2 - y^4 - x^2$. Complete the following:

   a. Use the Second Derivative test to find all extreme values and/or saddle points of f.

   b. Give the equation of the tangent plane of f at $(\frac{1}{2}, \frac{1}{2})$.

   c. In which direction would a drop of water start to slide, if the drop was placed on the surface of f at $(\frac{1}{2}, \frac{1}{2}, \frac{-1}{16})$?

   d. If $x(u, v) = u\cos(v)$ and $y(u, v) = v\sin(u)$, calculate $\frac{\partial f}{\partial u}$

2. Let $u = < 1, 0, -\sqrt{3} >$, $v = < -1, 0, -\sqrt{3} >$, and $w = < 2, 2, 1 >$ be vectors in $\mathbb{R}^3$, and let $\mathcal{P}$ be the parallelepiped determind by u, v, and w. Compute the following:

   a. $proj_v(u)$

   b. The angle between u and v

   c. The volume of $\mathcal{P}$.

3. Let $r(t) =< \sqrt{2}t, e^t, e^{-t} >$ for $0 \leq t \leq 1$ be the path of a particle in $\mathbb{R}^3$. Compute the following:

   a. The equation of the line tangent to $r(t)$ at $t = \frac{1}{2}$

   b. The unit tangent vector for $r(t)$.

   c. The arc legth of $r(t)$.

   d. The normal vector for $r(t)$.

4. Compute $\displaystyle\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}} dxdydz.$

Skip this problem.

5. Use the change of variables $u = x+y$ and $v = y-2x$ to evaluate $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dydx$

6. Let $\mathbf{F}(x, y) = y^3\mathbf{i} + (x^3 + 3xy^2)\mathbf{j}$ be a vector field, and let $\mathcal{R}$ be the oriented region in the first quadrant enclosed by the counterclockwise boundary curves $y = x^3$ and $y = x$.

   a. Show $\mathbf{F}$ is not conservative.

   b. Find a piecewise parameterization for $\mathcal{C}$ and calculate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ directly.

   c. Calculate $\displaystyle\int\int_{\mathcal{R}} curl(\mathbf{F}) \cdot \mathbf{k}\, dA.$

7. Let $\mathbf{F}(x, y, z) = (\frac{y^2}{z})\mathbf{i} + (\frac{2xy}{z})\mathbf{j} - (\frac{xy^2}{z^2})\mathbf{k}$ be a vector field in $\mathbb{R}^3$.

   a. Show that $\mathbf{F}$ is conservative and find a potential function f.

   b. Use the Fundamental Theorem of Line Integrals to evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, where $\mathcal{C}$ is the portion of the conical helix $x = t\cos(t)$, $y = t\sin(t)$, and $z = t$ for $\pi \leq t \leq \frac{3\pi}{2}$.

① a) $f_x = -2x$ $\quad f_y = 2y - 4y^3$

$-2x = 0$

$x = 0$

$2y - 4y^3 = 0$

$2y(1 - 2y^2) = 0$

$2y = 0 \qquad 1 - 2y^2 = 0$

$y = 0 \qquad 2y^2 = 1$

$y^2 = 1/2$

$y = \pm\sqrt{1/2} = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$

Critical Points

$(0,0), \ (0, \frac{\sqrt{2}}{2}), \ (0, -\frac{\sqrt{2}}{2})$

$f_{xx} = -2 \qquad f_{yy} = 2 - 12y^2 \qquad f_{xy} = 0$

$D(0,0) = (-2)(2 - 12(0)^2) - 0 = (-2)(2) = -4 < 0 \Rightarrow$ Saddle point

$D(0, \frac{\sqrt{2}}{2}) = (-2)(2 - 12(\frac{\sqrt{2}}{2})^2) - 0 = (-2)(-4) = 8 > 0$ & $f_{xx} < 0$

$\Rightarrow$ Local max

$D(0, -\frac{\sqrt{2}}{2}) = (-2)(2 - 12(-\frac{\sqrt{2}}{2})^2) - 0 = (-2)(-4) = 8 > 0$ & $f_{xx} < 0$

$\Rightarrow$ local max

Extreme Values

$f(0, \frac{\sqrt{2}}{2}) = (\frac{\sqrt{2}}{2})^2 - (\frac{\sqrt{2}}{2})^4 - 0^2 = \frac{1}{4}$

$f(0, -\frac{\sqrt{2}}{2}) = (-\frac{\sqrt{2}}{2})^2 - (-\frac{\sqrt{2}}{2})^4 - 0^2 = \frac{1}{4}$

Conclusion:

Saddle point at $(0,0)$.
Max value is $\frac{1}{4}$ at $(0, \frac{\sqrt{2}}{2})$.
Max value is $\frac{1}{4}$ at $(0, -\frac{\sqrt{2}}{2})$.

1 b) $f\left(\frac{1}{2},\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^2 = \frac{1}{4} - \frac{1}{16} - \frac{1}{4} = -\frac{1}{16}$

Pt $\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{16}\right)$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z + \frac{1}{16} = (-2)\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \left[2\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right)^3\right]\left(y - \frac{1}{2}\right)$$

$$z + \frac{1}{16} = (-1)\left(x - \frac{1}{2}\right) + \left(1 - \frac{1}{2}\right)\left(y - \frac{1}{2}\right)$$

$$z + \frac{1}{16} = -x + \frac{1}{2} + \frac{1}{2}y - \frac{1}{4}$$

$$z + \frac{1}{16} = -x + \frac{1}{2}y + \frac{1}{4}$$

$$\boxed{z = -x + \frac{1}{2}y + \frac{3}{16}}$$

1 c) DIRECTION OF

GREATEST DECREASE : $-\nabla f(x,y) = -\left[-2x\,\hat{\imath} + (2y - 4y^3)\,\hat{\jmath}\right]$

So $-\nabla f\left(\frac{1}{2}, \frac{1}{2}\right) = -\left[-\hat{\imath} + \left(1 - \frac{1}{2}\right)\hat{\jmath}\right]$

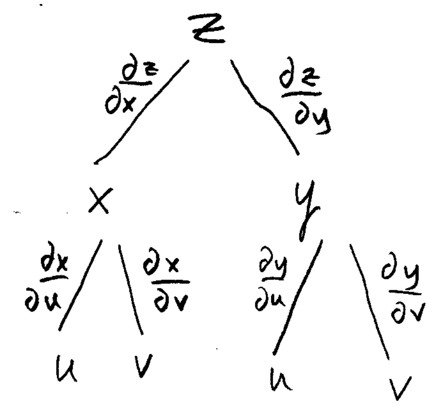
$$= \boxed{\hat{\imath} - \frac{1}{2}\hat{\jmath}}$$

① (d) <u>Use Chain Rule</u>

$f(x,y) = y^2 - y^4 - x^2$. Let $z = f(x,y) = y^2 - y^4 - x^2$

$x(u,v) = u\cos v$ and $y(u,v) = v\sin u$

$$\frac{\partial f}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$



$$\frac{\partial f}{\partial u} = [-2x][\cos v] + [2y - 4y^3][v\cos u]$$

$$= [-2(u\cos v)][\cos v] + [2(v\sin u) - 4(v\sin u)^3][v\cos u]$$

$$= \boxed{-2u\cos^2 v + 2v^2\sin u\cos u - 4v^4\sin^3 u\cos u}$$

(2)

a) $$\text{proj}_v(u) = \frac{\langle -1, 0, -\sqrt{3} \rangle \cdot \langle 1, 0, -\sqrt{3} \rangle}{|\langle -1, 0, -\sqrt{3} \rangle|^2} \langle -1, 0, -\sqrt{3} \rangle$$

$$= \frac{-1+0+3}{\left(\sqrt{1+0+3}\right)^2} \langle -1, 0, -\sqrt{3} \rangle$$

$$= \frac{2}{4} \langle -1, 0, -\sqrt{3} \rangle = \boxed{\left\langle -\frac{1}{2}, 0, -\frac{\sqrt{3}}{2} \right\rangle}$$

b) $$\cos\theta = \frac{\langle 1, 0, -\sqrt{3} \rangle \cdot \langle -1, 0, -\sqrt{3} \rangle}{|\langle 1, 0, -\sqrt{3} \rangle| \; |\langle -1, 0, -\sqrt{3} \rangle|} = \frac{-1+0+3}{\sqrt{1+0+3}\sqrt{1+0+3}} = \frac{2}{4} = \frac{1}{2}$$

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right) \Rightarrow \boxed{\theta = \frac{\pi}{3}}$$

c)
$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 1 & 0 & -\sqrt{3} \\ -1 & 0 & -\sqrt{3} \\ 2 & 2 & 1 \end{vmatrix} = 1 \begin{vmatrix} 0 & -\sqrt{3} \\ 2 & 1 \end{vmatrix} - 0 \begin{vmatrix} -1 & -\sqrt{3} \\ 2 & 1 \end{vmatrix} + (-\sqrt{3}) \begin{vmatrix} -1 & 0 \\ 2 & 2 \end{vmatrix}$$

$$= 1(2\sqrt{3}) - 0 - \sqrt{3}(-2) = 4\sqrt{3}$$

$$\text{VOLUME} = |\vec{u} \cdot (\vec{v} \times \vec{w})| = |4\sqrt{3}| = \boxed{4\sqrt{3}}$$

3. a) $\vec{r}\left(\frac{1}{2}\right) = \langle \sqrt{2}\cdot\frac{1}{2}, e^{1/2}, e^{-1/2} \rangle = \langle \frac{\sqrt{2}}{2}, \sqrt{e}, \frac{1}{\sqrt{e}} \rangle$

$\vec{r}\,'(t) = \langle \sqrt{2}, e^{t}, -e^{-t} \rangle$

$\vec{r}\,'\left(\frac{1}{2}\right) = \langle \sqrt{2}, e^{1/2}, -e^{-1/2} \rangle = \langle \sqrt{2}, \sqrt{e}, \frac{-1}{\sqrt{e}} \rangle$

$$\boxed{x = \frac{\sqrt{2}}{2} + \sqrt{2}\,t,\ y = \sqrt{e} + \sqrt{e}\,t,\ z = \frac{1}{\sqrt{e}} - \frac{1}{\sqrt{e}}\,t}$$

b) $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle \sqrt{2}, e^{t}, -e^{-t} \rangle}{\sqrt{2 + e^{2t} + e^{-2t}}} = \frac{\langle \sqrt{2}, e^{t}, -e^{-t} \rangle}{\sqrt{(e^{t} + e^{-t})^2}}$

$$= \boxed{\frac{\langle \sqrt{2}, e^{t}, -e^{-t} \rangle}{e^{t} + e^{-t}}}$$

c) $L = \int_0^1 |\langle \sqrt{2}, e^{t}, -e^{-t} \rangle|\, dt$

$= \int_0^1 \sqrt{2 + e^{2t} + e^{-2t}}\, dt$

$= \int_0^1 \sqrt{(e^{t} + e^{-t})^2}\, dt$

$= \int_0^1 (e^{t} + e^{-t})\, dt$

$= \left[e^{t} - e^{-t} \right|_0^1$

$= (e^{1} - e^{-1}) - (\cancel{e^{0}} - \cancel{e^{0}})$

$= \boxed{e - \frac{1}{e}}$

③(d)

Note: Problem asks for the normal vector not the unit normal vector. Using $\vec{T}(t)$ from part (b):

$$\vec{T}(t) = \frac{\langle \sqrt{2}, e^t, -e^{-t} \rangle}{e^t + e^{-t}} = \frac{(e^t)\langle \sqrt{2}, e^t, -e^{-t} \rangle}{(e^t)(e^t + e^{-t})} = \frac{\langle \sqrt{2}e^t, e^{2t}, -1 \rangle}{e^{2t}+1}$$

$$= (e^{2t}+1)^{-1} \langle \sqrt{2}e^t, e^{2t}, -1 \rangle$$

$$\vec{T}'(t) = (e^{2t}+1)^{-1} \langle \sqrt{2}e^t, 2e^{2t}, 0 \rangle + (-1)(e^{2t}+1)^{-2}(2e^{2t})\langle \sqrt{2}e^t, e^{2t}, -1$$

$$= \boxed{\frac{1}{e^{2t}+1} \langle \sqrt{2}e^t, 2e^{2t}, 0 \rangle - \frac{2e^{2t}}{(e^{2t}+1)^2} \langle \sqrt{2}e^t, e^{2t}, -1 \rangle}$$

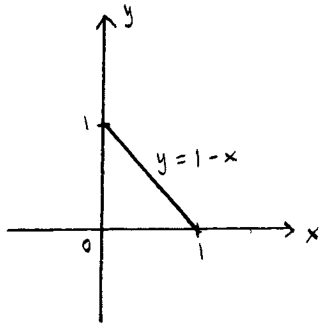
④ Skip this problem!

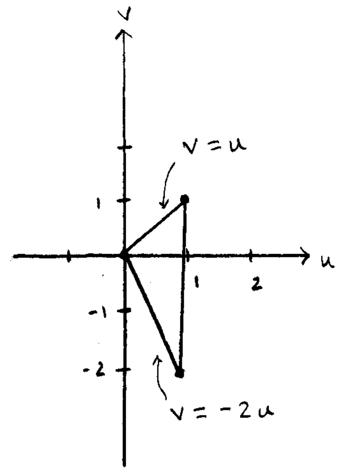
(5) $\displaystyle\int_0^1\int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3$$

Pts in (x,y)
$(0,1)$
$(0,0)$
$(1,0)$

$u = x + y \quad v = y - 2x$





Pts in (u,v)
$(1,1)$
$(0,0)$
$(1,-2)$

$$\frac{1}{3}\int_0^1\int_{-2u}^{u} \sqrt{u}\,(v)^2\,dv\,du$$

Inside integral $\Rightarrow \displaystyle\int_{-2u}^{u} \sqrt{u}\,(v^2)\,dv\,du$

$$= \sqrt{u}\left(\tfrac{1}{3}v^3\right)\Big|_{v=-2u}^{v=u}$$

$$= \sqrt{u}\left[\tfrac{1}{3}u^3 - \tfrac{1}{3}(-2u)^3\right]$$

$$= \tfrac{1}{3}u^{7/2} + \tfrac{8}{3}u^{7/2} = 3u^{7/2}$$

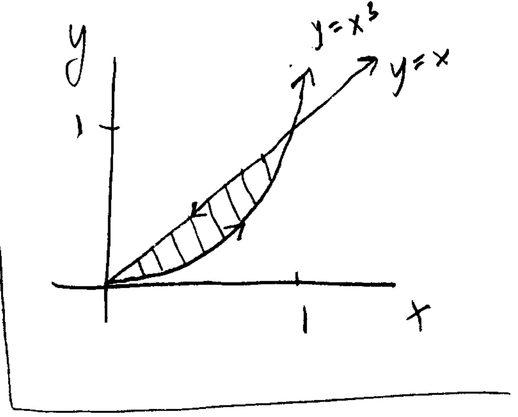
$$\frac{1}{3}\int_0^1 3u^{7/2}\,du = \int_0^1 u^{7/2}\,du$$

$$= \frac{2}{9}u^{9/2}\Big|_0^1$$

$$= \frac{2}{9}(1)^{9/2} - \frac{2}{9}(0)^{9/2}$$

$$= \boxed{\frac{2}{9}}$$

6 (a) $P = y^3$ and $Q = x^3 + 3xy^2$



$\dfrac{\partial P}{\partial y} = 3y^2$ and $\dfrac{\partial Q}{\partial x} = 3x^2 + 3y^2$

Not conservative since $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$.

(b) C_1: $y = x^3 \Rightarrow x = t,\ y = t^3 \Rightarrow \vec{r}_1(t) = t\vec{i} + t^3\vec{j},\ 0 \le t \le 1$

C_2: $y = x \Rightarrow x = t,\ y = t \Rightarrow \vec{r}_2(t) = t\vec{i} + t\vec{j},\ 1 \le t \le 0$

(since counter-clockwise)

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}(\vec{r}(t))\cdot\vec{r}_1{}'(t)\,dt + \int_{C_2} \vec{F}(\vec{r}(t))\cdot\vec{r}_2{}'(t)\,dt$$

$$= \int_0^1 \left[(t^3)^3\vec{i} + \left(t^3 + 3t(t^3)^2\right)\vec{j}\right]\cdot\left[\vec{i} + 3t^2\vec{j}\right]dt$$

$$+ \int_1^0 \left[t^3\vec{i} + (t^3 + 3tt^2)\vec{j}\right]\cdot\left[\vec{i} + \vec{j}\right]dt$$

$$= \int_0^1 \left[t^9\vec{i} + (t^3 + 3t^7)\vec{j}\right]\cdot\left[\vec{i} + 3t^2\vec{j}\right]dt$$

$$+ \int_1^0 \left[t^3\vec{i} + (t^3 + 3t^3)\vec{j}\right]\cdot\left[\vec{i} + \vec{j}\right]dt$$

$$= \int_0^1 \left[(t^9)(1) + (t^3 + 3t^7)(3t^2)\right]dt + \int_1^0 \left[(t^3)(1) + (t^3 + 3t^3)(1)\right]dt$$

$$= \int_0^1 \left[t^9 + 3t^5 + 9t^9\right]dt + \int_1^0 \left[5t^3\right]dt = \int_0^1 \left[10t^9 + 3t^5\right]dt + \int_1^0 5t^3\,dt$$

$$= \left[t^{10} + \tfrac{1}{2}t^6\right]_0^1 + \left[\tfrac{5}{4}t^4\right]_1^0 = \left(1 + \tfrac{1}{2}\right) + \left(0 - \tfrac{5}{4}\right) = \boxed{\tfrac{1}{4}}$$

⑥ (c)

$$\iint_R \operatorname{curl}(\vec{F})\cdot\vec{k}\, dA = \oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy$$

where $P = y^3$ and $Q = x^3 + 3xy^2$

Using Green's Theorem, we have:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_{x^3}^{x} \left[\left(3x^2 + 3y^2\right) - 3y^2\right] dy\,dx$$

$$= \int_0^1 \int_{x^3}^{x} 3x^2\, dy\,dx = \int_0^1 \left[3x^2 y\right]_{y=x^3}^{y=x} dx = \int_0^1 \left[3x^3 - 3x^5\right] dx$$

$$= \left[\frac{3}{4}x^4 - \frac{1}{2}x^6\right]_0^1 = \frac{3}{4} - \frac{1}{2} = \boxed{\frac{1}{4}}$$

7

(a) $\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y^2}{z} & \frac{2xy}{z} & -\frac{xy^2}{z^2} \end{vmatrix} = \left(\frac{\partial}{\partial y}\left[\frac{-xy^2}{z^2}\right] - \frac{\partial}{\partial z}\left[\frac{2xy}{z}\right]\right)\vec{i}$

$$- \left(\frac{\partial}{\partial x}\left[\frac{-xy^2}{z^2}\right] - \frac{\partial}{\partial z}\left[\frac{y^2}{z}\right]\right)\vec{j}$$

$$+ \left(\frac{\partial}{\partial x}\left[\frac{2xy}{z}\right] - \frac{\partial}{\partial y}\left[\frac{y^2}{z}\right]\right)\vec{k}$$

$$= \left(\frac{-2xy}{z^2} - \frac{-2xy}{z^2}\right)\vec{i} - \left(\frac{-y^2}{z^2} - \frac{-y^2}{z^2}\right)\vec{j} + \left(\frac{2y}{z} - \frac{2y}{z}\right)\vec{k} = \vec{0}$$

By Theorem $\boxed{4}$, since $\text{curl}\,\vec{F} = \vec{0}$, then $\boxed{\vec{F} \text{ is conservative}}$.

Find potential function:

$f_x(x,y,z) = \frac{y^2}{z}$ $\quad$ $f_y(x,y,z) = \frac{2xy}{z}$ $\quad$ $f_z(x,y,z) = \frac{-xy^2}{z^2}$

$f(x,y,z) = \int f_x(x,y,z)\,dx = \int \frac{y^2}{z}\,dx = \frac{xy^2}{z} + g(y,z)$.

So $f_y(x,y,z) = \frac{\partial}{\partial y}\left[\frac{xy^2}{z} + g(y,z)\right] = \frac{2xy}{z} + g_y(y,z)$ which gives us

$\frac{2xy}{z} = \frac{2xy}{z} + g_y(y,z) \Rightarrow g_y(y,z) = 0$ and so $g(y,z) = \int g_y(y,z)\,dy = \int 0\,dy = 0 + h(z)$
$= h(z)$.

Now we have from above, $f(x,y,z) = \frac{xy^2}{z} + h(z)$.

Then $f_z(x,y,z) = \frac{\partial}{\partial z}\left[\frac{xy^2}{z} + h(z)\right] = \frac{-xy^2}{z^2} + h'(z)$ which gives us

$\frac{-xy^2}{z^2} = \frac{-xy^2}{z^2} + h'(z) \Rightarrow h'(z) = 0$ and so $h(z) = \int h'(z)\,dz = \int 0\,dz = 0 + C = K$

So our potential function is $\boxed{f(x,y,z) = \frac{xy^2}{z} + K}$

⑦ (b) Potential function f from part (a) is

$$f(x,y,z) = \frac{xy^2}{z} + K$$

Find end points of curve C:

$x = t\cos t$, $y = t\sin t$, $z = t$, for $\pi \le t \le \frac{3\pi}{2}$

Using $t = \pi$ we get

$$\left.\begin{array}{lll} x = \pi\cos\pi, & y = \pi\sin\pi, & z = \pi \\ x = -\pi, & y = 0, & z = \pi \end{array}\right\} \quad (-\pi, 0, \pi)$$

Using $t = \frac{3\pi}{2}$ we get

$$\left.\begin{array}{lll} x = \frac{3\pi}{2}\cos\frac{3\pi}{2}, & y = \frac{3\pi}{2}\sin\frac{3\pi}{2}, & z = \frac{3\pi}{2} \\ x = 0, & y = -\frac{3\pi}{2}, & z = \frac{3\pi}{2} \end{array}\right\} \quad \left(0, -\frac{3\pi}{2}, \frac{3\pi}{2}\right)$$

Use Fundamental Theorem of Line Integrals:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = f(x_2,y_2,z_2) - f(x_1,y_1,z_1)$$

$$= f\left(0, -\frac{3\pi}{2}, \frac{3\pi}{2}\right) - f(-\pi, 0, \pi)$$

$$= \frac{0\left(-\frac{3\pi}{2}\right)^2}{\frac{3\pi}{2}} - \frac{(-\pi)(0)^2}{\pi} = \boxed{0}$$