Typo #9|11

Math 184 Exam 3 SHOW ALL WORK Name ____kry____

1. Consider the transformation $T: P_n \to P_{n+1}$ defined by $T(p(x)) = xp(x) - 2p(x)$. Is T a linear transformation? Prove that your answer is correct.

(1) $T(p(x)+q(x)) = x[p(x)+q(x)] - 2[p(x)+q(x)]$

$= [xp(x) - 2p(x)] + [xq(x) - 2q(x)] = T(p(x)) + T(q(x))$

(2) $T[c \cdot p(x)] = x \cdot cp(x) - 2cp(x)$

$= c[xp(x) - 2p(x)] = c \cdot T(p(x))$

so yes

2. Consider the linear transformation $T: R^3 \to R^3$ where $T(\vec{v}) = A\vec{v}$. Find a basis for the kernel of this transformation, given that $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \\ 0 & 4 & 7 \end{bmatrix}$.

$\ker(T) = N(A)$

$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 4 & 7 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 1\frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

A basis for $\ker(T) = \begin{bmatrix} \frac{1}{2} \\ -1\frac{3}{4} \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ -7 \\ 4 \end{bmatrix}$

$\lambda = 3$ (mult. 2)

3. Is the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & -1 & 2 \end{bmatrix}$ diagonalizable? Explain.

$\lambda I - A = \begin{bmatrix} \lambda-3 & 0 & 0 \\ -1 & \lambda-3 & 0 \\ -1 & 1 & \lambda-2 \end{bmatrix}$

$\lambda = 3$: $\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to$ evector is $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

only 1 evector for $\lambda = 3$ (mult. 2) so no

$$\begin{vmatrix} \lambda-1 & 6 \\ -4 & \lambda-3 \end{vmatrix} = 0 \rightarrow \lambda^2 - 4\lambda - 21 = 0 \quad (\lambda-7)(\lambda+3)=0$$

$$\lambda = 7, -3$$

4. Find all eigenvalues and bases for the corresponding eigenspaces for $A = \begin{bmatrix} 1 & 6 \\ 4 & 3 \end{bmatrix}$.

$$\lambda = 7: \left[\begin{array}{cc|c} 6 & -6 & 0 \\ -4 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = -3: \left[\begin{array}{cc|c} -4 & -6 & 0 \\ -4 & -6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \begin{bmatrix} -3 \\ 2 \end{bmatrix} \text{ or } \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$$

5. Use your answer to Problem 4 to solve the system $\vec{Y}' = A\vec{Y}$, where A is the matrix in Problem 4.

$$\vec{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7x} + c_2 \begin{bmatrix} -3 \\ 2 \end{bmatrix} e^{-3x}$$

6. Suppose 3x3 matrix A has eigenvalues $\lambda = 4$(multiplicity two) and $\lambda = 0$, with $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ a basis for E_4 and $\begin{bmatrix} 7 \\ 6 \\ 3 \end{bmatrix}$ a basis for E_0. Find the matrix P that you would use to diagonalize A, and also find the diagonal matrix D associated with A.

$$P = \begin{bmatrix} 1 & 0 & 7 \\ 2 & 2 & 6 \\ 3 & 1 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

7. Solve the system $\vec{Y}' = A\vec{Y}$, where A is the matrix given in problem 6.

$$\vec{Y} = c_1\begin{bmatrix}1\\2\\3\end{bmatrix}e^{4x} + c_2\begin{bmatrix}0\\2\\1\end{bmatrix}e^{4x} + c_3\begin{bmatrix}7\\6\\3\end{bmatrix}$$

8. Solve the system: $\begin{aligned} y_1' &= 3y_1 - 18y_2 \\ y_2' &= 2y_1 - 9y_2 \end{aligned}$

$$A = \begin{bmatrix}3 & -18\\2 & -9\end{bmatrix} \qquad \lambda^2 + 6\lambda + 9 = 0$$

$$\lambda I - A = \begin{bmatrix}\lambda-3 & 18\\-2 & \lambda+9\end{bmatrix} \qquad \lambda = -3 \text{ (mult. 2)}$$

$$\lambda = -3: \left[\begin{array}{cc|c}-6 & 18 & 0\\-2 & 6 & 0\end{array}\right] \to \left[\begin{array}{cc|c}1 & -3 & 0\\0 & 0 & 0\end{array}\right] \to t\begin{bmatrix}3\\1\end{bmatrix}$$

$$\left[\begin{array}{cc|c}6 & -18 & 3\\2 & -6 & 1\end{array}\right] \to \left[\begin{array}{cc|c}1 & -3 & \frac{1}{2}\\0 & 0 & 0\end{array}\right] \to \vec{p} = \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix} \text{ or } \begin{bmatrix}\frac{7}{2}\\1\end{bmatrix}$$

$$\vec{Y} = c_1\begin{bmatrix}3\\1\end{bmatrix}e^{-3x} + c_2\left(\begin{bmatrix}3\\1\end{bmatrix}xe^{-3x} + \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}e^{-3x}\right)$$

9. Consider the system: $\begin{aligned} y_1' &= 3y_1 - 18y_2 + \sin(x) \\ y_2' &= 2y_1 - 9y_2 - \ln(x) \end{aligned}$. Use your answer to Problem 8 to help find the matrix M and vector G that you would use to solve the system. Do not solve.

$$M = \begin{bmatrix}3e^{-3x} & 3xe^{-3x} + \frac{1}{2}e^{-3x}\\ e^{-3x} & xe^{-3x}\end{bmatrix} \qquad \vec{G} = \begin{bmatrix}\sin x\\ -\ln x\end{bmatrix}$$

10. Consider the linear transformation $T: P_1 \to P_1$ where $T(ax+b) = 2ax + a - 2b$. Find the eigenvalues and bases for the corresponding eigenspaces of this linear transformation, using $A = [T]_\alpha^\alpha$ where $\alpha = \{x, 1\}$.

$T(1\cdot x + 0\cdot 1) = 2\cdot x + 1\cdot 1$

$T(0\cdot x + 1\cdot 1) = 0\cdot x - 2\cdot 1$

$\to A = \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$ $\lambda = 2, -2$

$\lambda = 2$: $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & 4 & 0 \end{array}\right] \to \begin{bmatrix} 4 \\ 1 \end{bmatrix} \to 4x+1$

$\lambda = -2$: $\begin{bmatrix} -4 & 0 \\ -1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to 1$

scores

8	2 3 1 8
7	7
6	8
5	7 7 0
4	4
3	1

11. Let $T_{\sin x}(f(x)) = \sin x \cdot f(x)$ and let D be the differential operator. Find $(D^2 T_{\sin x})(e^x)$.

$\frac{d}{dx}\left(\frac{d}{dx}\left(\sin x e^x\right)\right)$

$= \frac{d}{dx}\left(\sin x e^x + \cos x e^x\right)$

$= \left(\sin x e^x + \cos x e^x - \sin x e^x + \cos x e^x\right) = 2\cos x e^x$